Nonexistence theorems for traversable wormholes

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Abstract

Gauss–Bonnet formula is used to derive a new and simple theorem of nonexistence of vacuum static nonsingular lorentzian wormholes. We also derive simple proofs for the nonexistence of lorentzian wormhole solutions for some classes of static matter such as, for instance, real scalar fields with a generic potential obeying $\phi V'(\phi) \geq 0$ and massless fermions fields.

A lorentzian wormhole is a solution of Einstein equations with asymptotically flat regions connected by intermediary "throats". These solutions have received considerable attention since Morris, Thorne and Yurtsever [1,2] discussed the possibility of traversing them and their connection with time machines. (For a review, see [3].) Euclidean wormholes, *i.e.*, wormhole solutions of Einstein equations with signature (+,+,+,+), have also been intensively studied in connection with the cosmological constant problem [4].

Originally, the analysis of lorentzian wormholes was restricted to the static spherically symmetrical case [1,2]. Birkhoff's theorem assures that the only vacuum spherically symmetrical lorentzian wormhole is the maximally extended Schwarzschild solution. However, in this case the "throat"connecting the two asymptotically flat regions ($r \to \pm\infty$) is singular for $r = 0$, and hence it is not traversable [1,2,5]. We recall that a lorentzian wormhole solution is said to be traversable [1,2] if it does not contain horizons that prevent the crossing of the "throats" and if an observer crossing them does not experience strong tidal forces. In order to obtain a solution with a traversable wormhole, we are enforced to accept the presence of matter and/or to give up of the spherical symmetry. A list of recent solutions includes stationary [6] and static axisymmetric electrovac [7] cases, and solutions in Brans-Dicke [8], Kaluza-Klein [9], Einstein–Gauss–Bonnet [10], Einstein–Cartan [11] and nonsymmetric field [12] theories. Some dynamical, *i.e.*, time-dependent, solutions have also been proposed [13].

For the Euclidean case, there are some theorems about the nonexistence of vacuum, *i.e.*, Ricci flat, wormholes. In [14], it was presented a theorem stating that any asymptotically flat, nonsingular, Ricci flat metric in $R^4 - \{N \ \text{ points}\}$ is flat. The proof uses some topological invariants of four-dimensional manifolds that can be expressed by means of integrals of curvature invariants, *viz.* the signature $\tau(\mathcal{M})$ and the Euler characteristic $\chi(\mathcal{M})$. The latter can be expressed, for a closed orientable diffentiable manifold $\mathcal{M}$ of dimension $n = 2p$ endowed with a Riemannian metric g, by the Gauss-Bonnet formula as:

$$\int_{\mathcal{M}} \epsilon_{a_1 \cdots a_n} R^{a_1 a_2} \cdots R^{a_{n-1} a_n} = (-1)^{p-1} 2^{2p} \pi^p p! \chi(\mathcal{M}), \tag{1}$$

where ${R_a}^b$ is the curvature 2-form of g. The signature can also be expressed by an integral

of curvature invariants. (See [14] for references).

One can consider a four-dimensional manifold with $N+1$ asymptotically flat regions as being topologically equivalent to R^4 with N points removed; one to each additional asymptotic region. Hence, the result of [14] rules out the existence of vacuum nonsingular euclidean wormhole solutions. This result was extended to the non-empty case in [15], where it is shown that there is no nonsingular euclidean wormhole satisfying Einstein equations with conformal invariant matter fields obeying appropriate falloff conditions in the asymptotic regions.

For the lorentzian case, there are no equivalent theorems in the literature. We remind a long standing theorem due to Lichnerowicz [16], which states that any stationary, complete, asymptotically flat, and Ricci-flat lorentzian metric in R^4 is flat. However, this theorem cannot be extended to the case of many asymptotically flat regions ($R^4 - \{N \text{ points}\}$). The purpose of the present work is to contribute to fill this gap with the following theorem:

> *Any asymptotically flat, static, nonsingular, Ricci flat lorentzian metric in* $R^4 - \{N \text{ points}\}$ *is flat.*

We will show that for the non-empty case some nonexistence theorems can also be formulated. The approach to prove our main result, based on the Gauss-Bonnet formula, will be similar to the used in [14]. Although Gauss-Bonnet theorem is rather subtle for other signatures [17], we can use Chern's intrinsic proof [18] of the Gauss-Bonnet formula (1) with minor modifications.

Let us now briefly review some points of Chern's proof with relevance to our purposes. In [18], Chern showed that Ω can be written as $\Omega = d\Pi$ for a suitable $(n-1)$-form Π. We will consider here this problem for the case of an open four-dimensional manifolds with a lorentzian complete metric.

Be V a continuous unit timelike vector field ($V_a V^a = -1$). Note that contrary to the case of $\mathcal{M}$ closed, here there is no topological obstruction to the global existence of such a vector field. Let us introduce the vector-valued 1-form

$$\theta^a = DV^a = dV^a + \omega^a_b V^b, \tag{2}$$

where ω^a_b is the Levi-Civita connection 1-form. Due to that $V^a V_a = -1$, the 1-forms θ^a are linearly dependent, indeed $V_a \theta^a = 0$. From (2) we have

$$d\theta^a = \theta^b \wedge \omega^a_b + R_b{}^a V^b. \tag{3}$$

Now, consider the following 3-forms

$$\begin{aligned} \phi_0 &= \epsilon_{abcd} V^a \wedge \theta^b \wedge \theta^c \wedge \theta^d, \\ \phi_1 &= \epsilon_{abcd} V^a \wedge \theta^b \wedge R^{cd}. \end{aligned} \tag{4}$$

Using the linearly dependence of the θ^a, the Bianchi's identities, (2), and (3) we get

$$\begin{aligned} & d\phi_0 - 3\epsilon_{abcd} V^a \wedge V_f \wedge R^{fb} \wedge \theta^c \wedge \theta^d = \text{terms prop. to } \omega^a_b, \\ & d\phi_1 - \epsilon_{abcd} \left(\theta^a \wedge \theta^b \wedge R^{cd} + V^a \wedge V_f \wedge R^{fb} \wedge R^{cd} \right) = \text{terms prop. to } \omega^a_b. \end{aligned} \tag{5}$$

The left-handed sides of (5) are clearly intrinsic expressions, and, by choosing normal coordinates around an arbitrary point, we can easily show that they indeed vanish.

We can cast (5) in a more convenient form by using the generalized Kronecker delta

$$\delta^{a_1 \cdots a_n}_{b_1 \cdots b_n} = \begin{vmatrix} \delta^{a_1}_{b_1} & \cdots & \delta^{a_1}_{b_n} \\ \vdots & & \vdots \\ \delta^{a_n}_{b_1} & \cdots & \delta^{a_n}_{b_n} \end{vmatrix}. \tag{6}$$

For $n > 4$, $\delta^{a_1 \cdots a_n}_{b_1 \cdots b_n}$ vanishes identically. In particular, one has

$$\delta^{a'b'c'd'f'}_{a\,b\,c\,d\,f} = \delta^{f'}_a \delta^{a'b'c'd'}_{b\,c\,d\,f} - \delta^{f'}_b \delta^{a'b'c'd'}_{a\,c\,d\,f} + \delta^{f'}_c \delta^{a'b'c'd'}_{a\,b\,d\,f} - \delta^{f'}_d \delta^{a'b'c'd'}_{a\,b\,c\,f} + \delta^{f'}_f \delta^{a'b'c'd'}_{a\,b\,c\,d} = 0 \tag{7}$$

Using that $V_a V^a = -1$, the linearly dependence of θ^a, and (7) we obtain

$$\begin{aligned} \epsilon_{abcd} V^a \wedge V_f \wedge R^{fb} \wedge \theta^c \wedge \theta^d &= -\frac{1}{2} \epsilon_{abcd} R^{ab} \wedge \theta^c \wedge \theta^d, \\ \epsilon_{abcd} V^a \wedge V_f \wedge R^{fb} \wedge R^{cd} &= -\frac{1}{4} \epsilon_{abcd} R^{ab} \wedge R^{cd}. \end{aligned} \tag{8}$$

From (8) we finally get $\Omega = \epsilon_{abcd} R^{ab} \wedge R^{cd} = d\Pi$, where

$$\Pi = -4\left(\phi_1 - \frac{2}{3}\phi_0\right) = -4\epsilon_{abcd}V^a \wedge \theta^b \wedge \left(R^{cd} - \frac{2}{3}\theta^c \wedge \theta^d\right). \tag{9}$$

The expression (9) is essentially the result of Chern. The difference is that in the case of closed manifolds, Π is not defined everywhere. In general, continuous vector fields have some singular points in a closed manifold, and thus we cannot construct a vector field with $V_aV^a \neq 0$ everywhere. As already said, in the present case there is no topological obstruction to the existence of a nowhere vanishing vector field.

We can now prove our result. For static, Ricci-flat lorentzian metrics Ω is non negative. To see it, first note that with $R_{ab} = 0$ one has

$$\Omega = \frac{1}{6}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right)\nu = \frac{1}{6}R_{abcd}R^{abcd}\nu, \tag{10}$$

where ν is the standard volume form in R^4. Consider now normal coordinates around an arbitrary point P. We have

$$\Omega_P = \frac{\nu_P}{6}\left(-4\sum_{\alpha\beta\gamma}\left(R_{1\alpha\beta\gamma}\right)^2_P + 4\sum_{\alpha\beta}\left(R_{1\alpha1\beta}\right)^2_P + \sum_{\alpha\beta\gamma\delta}\left(R_{\alpha\beta\gamma\delta}\right)^2_P\right). \tag{11}$$

Hereafter, Roman and Greek indices run, respectively, over $\{1,2,3,4\}$ and $\{2,3,4\}$. Due to the assumption of a nonsingular g, we have $\Omega < \infty$. The hypothesis that g is static assures that there is always a coordinate system $\{x^a\}$ for which

$$g_{ab} = \begin{pmatrix} -\Delta & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}, \tag{12}$$

where $\Delta > 0$ and g_{ab} does not depend on x^1. One can check that for metrics of the type (12), $R_{1\alpha\beta\gamma} = 0$, and consequently $\Omega \geq 0$. Moreover, the equality holds only if $R_{abcd} = 0$. Thus, in the case of $\int_{\mathcal{X}} \Omega = 0$, we have that $R_{abcd} = 0$ in $\mathcal{X}$. This will be the strategy of our proof. The assumption of an asymptotically flat metric guarantees that for each asymptotic region one has

$$\begin{aligned} \lim_{r\to\infty} \Delta - 1 &= O^\infty\left(\frac{1}{r^{m+\varepsilon}}\right), \\ \lim_{r\to\infty} h_{\alpha\beta} - \delta_{\alpha\beta} &= O^\infty\left(\frac{1}{r^{m+\varepsilon}}\right). \end{aligned} \tag{13}$$

for some $m \in \{0, 1, 2, \ldots\}$ and $0 < \varepsilon \leq 1$, where $r^2 = x^\alpha x^\beta \delta_{\alpha\beta}$. If $F = O^\infty(f)$ it means that $F = O(|f|)$, $F' = O(|f'|)$, and so on. We can construct the 3-form Π starting from the timelike unit vector $V = \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial x^1}$. From (13), we obtain the following expression, valid for each asymptotic region,

$$\lim_{r \to \infty} \Pi = O^\infty \left(\frac{1}{r^{m+3+\varepsilon}} \right). \tag{14}$$

Integrating Ω over $\mathcal{M} = R^4 - \{N \text{ points}\} \approx S^4 - \{(N+1) \text{ points}\}$ and with the assumption of a nonsingular g one has

$$\int_{\mathcal{M}} \Omega = \sum_{i=1}^{N+1} \int_{\partial_i \mathcal{M}} \Pi, \tag{15}$$

where the boundaries $\partial_i \mathcal{M}$ correspond to the asymptotic regions. For each of these regions, with the asymptotic conditions (13), we have

$$\lim_{r \to \infty} \mu \left(\partial_i \mathcal{M} \right) = O^\infty \left(r^3 \right), \tag{16}$$

where $\mu \left(\partial_i \mathcal{M} \right)$ denotes the measure of the boundary $\partial_i \mathcal{M}$. From (14) and (16) we have finally that the right handed side of (15) vanishes, establishing our result.$\square$

It is shown in [15] that, for the euclidean case, sometimes the matter equations themselves can be used to rule out non-vacuum wormholes. The same arguments can be applied here for some static matter fields. Let us take, for instance, a real scalar field $\phi(x)$ with a potential obeying $\phi V'(\phi) \geq 0$. The hypothesis of a nonsingular g requires that ϕ and $\partial_a \phi$ be smooth and bounded on $\mathcal{M}$. The corresponding equation in this case is

$$D_a D^a \phi = V'(\phi). \tag{17}$$

Multiplying by ϕ and integrating over $\mathcal{M}$ one obtains

$$\int_{\mathcal{M}} \left(g^{ab} (\partial_a \phi)(\partial_b \phi) + \phi V'(\phi) \right) \, d\mathrm{vol} - \sum_{i=1}^{N+1} \int_{\partial_i \mathcal{M}} \phi \partial_a \phi \, d\Sigma^a = 0. \tag{18}$$

With the assumption that ϕ obeys an asymptotic condition like

$$\lim_{r \to \infty} \phi = O^\infty \left(\frac{1}{m + \varepsilon} \right), \tag{19}$$

the boundary terms in (18) vanish. This assumption requires that $V'(\phi) = 0$ for $\phi = 0$. The first term in (18) is, for the static case, nonnegative, implying that ϕ must vanish in $\mathcal{M}$ if $\phi V'(\phi) \geq 0$. We make here the same remark done for the euclidean case [15]: if $V'(\phi) = 0$ for some $\phi \neq 0$, and ϕ assumes to a nonzero value in some of the asymptotic regions, the boundary term in (18) may not vanish in general. A nonexistence result holds also for conformally coupled massless fields. In this case, we have

$$D_a D^a \phi - \frac{R}{6}\phi = 0. \tag{20}$$

However, Einstein equations imply in this case that $R = 0$, and we get in fact a particular case of (18).

Another example of nonexistence of non-empty wormhole solutions is the case of massless fermions. In this case, the matter equations are

$$i\partial_a \gamma^a \psi = 0. \tag{21}$$

Applying the Dirac operator $i\partial_a\gamma^a$ again one gets a conformally coupled Klein-Gordon equation for each component of ψ

$$\left(D_a D^a - \frac{1}{6}R\right)\psi = 0. \tag{22}$$

Einstein equations also imply that $R = 0$ in this case. Also, we can redefine ψ in order to have only real components. As in the previous cases, provided that ψ obeys appropriate falloff conditions in the asymptotic regions, we conclude that there is no wormhole solution with massless fermions.

We finish noticing that, with the same procedure use here, we can show that in $R^4 - \{N \text{ points}\}$, any metric of signature $(-,-,+,+)$, Ricci-flat, asymptotically flat, static simultaneously with respect to two linearly independent time-like Killing vectors, is flat.

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